Last time:

$$\langle 0 | F(y) | \Psi_{\vec{p},\sigma,n} \rangle \neq 0 \Rightarrow q_{(n)} = q_F \qquad (1)$$

Consider now Dirac Lagrangian coupled to electromagnetic field:

$$\mathcal{L} = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$
$$+ \mathcal{L}_M\left(\Psi_e, [\partial_\mu - i q_e A_\mu]\Psi_e\right)$$

The current is given by

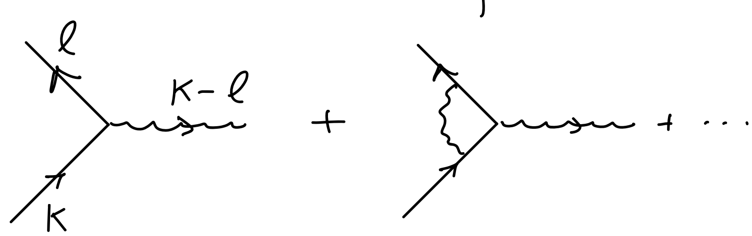
$$J^\mu = \frac{\delta \mathcal{L}_M}{\delta A_\mu}$$

Then the following Green's function in momentum space:

$$\int d^4x\, d^4y\, d^4z\, e^{-ip\cdot x} e^{-ik\cdot y} e^{+il\cdot z}$$
$$\times \langle 0 | T[J^\mu(x) \Psi_n(y) \bar{\Psi}_m(z)] | 0 \rangle$$

$$= -i(2\pi)^4 \underbrace{q}_{\text{electric charge}} S'_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,l)\, S'_{n'm}(l)\, \delta^{(4)}(p+k-l) \qquad (2)$$

corresponds to the vertex function:

$\Gamma^\mu =$ 

where S' is the renormalized Dirac propagator:

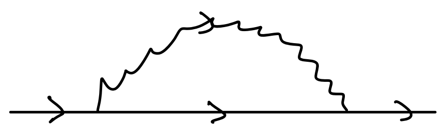
$$S'(k) = \frac{i}{\not{k} - m + i\varepsilon} + \frac{i}{\not{k} - m + i\varepsilon} \Sigma^*(\not{k}) \frac{i}{\not{k} - m + i\varepsilon}$$

$$+ \frac{i}{\not{k} - m + i\varepsilon} \Sigma^*(\not{k}) \frac{i}{\not{k} - m + i\varepsilon} \Sigma^*(\not{k}) \frac{i}{\not{k} - m + i\varepsilon}$$

$$+ \cdots$$

$$= \frac{i}{\not{k} - m - \Sigma^*(\not{k})}$$

and $\Sigma^*(\not{k})$ is computed from diagrams of the form:

 $+ \cdots$

S' has following operator expression

$$i(2\pi)^4 S'_{nm}(k)\, \delta^{(4)}(k-l)$$
$$= \int d^4y\, d^4z \langle 0 | T[\psi_n(y) \bar{\psi}_m(z)] | 0 \rangle e^{-ik\cdot y} e^{+il\cdot z} \qquad (3)$$

In the limit of no interactions:

$$iS'(k) \to i[\not{k} - m + i\varepsilon]^{-1}, \quad \Gamma^\mu(k,l) \to \gamma^\mu$$

We can derive relations between Γ^{μ} and S' by using the identity:

$$\frac{\partial}{\partial x^{\mu}} T\left[j^{\mu}(x) \psi_n(y) \bar{\psi}_m(z) \right]$$

$$= T\left[\partial_{\mu} j^{\mu}(x) \psi_n(y) \bar{\psi}_m(z) \right]$$

$$+ \delta(x^0 - y^0) \, T\left[[j^0(x), \psi_n(y)] \bar{\psi}_m(z) \right] \qquad (*)$$

$$+ \delta(x^0 - z^0) \, T\left[\psi_n(y) \, [j^0(x), \bar{\psi}_m(z)] \right]$$

where the delta functions arise from time-derivatives of step functions as follows:

$$T\left[j^{\mu}(x) \psi_n(y) \right] = \theta(x^0 - y^0) j^{\mu}(x) \psi_n(y) + \theta(y^0 - x^0) \psi_n(y) j^{\mu}(x)$$

$$\rightarrow \frac{\partial}{\partial x^{\mu}} T\left[j^{\mu}(x) \psi_n(y) \right]$$

$$= \delta(x^0 - y^0) j^0(x) \psi_n(y) - \delta(y^0 - x^0) \psi_n(y) j^{\mu}(x)$$

$$= \delta(x^0 - y^0) \left[j^0(x), \psi_n(y) \right]$$

and analogously for $\frac{\partial}{\partial x^{\mu}} T\left[j^{\mu}(x) \bar{\psi}_m(z) \right]$

Current conservation $\longrightarrow$ 1st term in (*) vanishes

the other two term are computed by using:

$$[j^0(\vec{x},t), \psi_n(\vec{y},t)] = -q\,\psi_n(\vec{y},t)\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$[j^0(\vec{x},t), \bar{\psi}_n(\vec{y},t)] = q\,\bar{\psi}(\vec{y},t)\,\delta^{(3)}(\vec{x}-\vec{y})$$

$\longrightarrow$ (*) becomes:

$$\frac{\partial}{\partial x^\mu} T[j^\mu(x)\,\psi_n(y)\,\bar{\psi}_m(z)]$$

$$= -q\,\delta^{(4)}(x-y)\,T[\psi_n(y)\,\bar{\psi}_m(z)]$$

$$+ q\,\delta^{(4)}(x-z)\,T[\psi_n(y)\,\bar{\psi}_m(z)]$$

Inserting into (2) gives

$$(l-k)_\mu\, S'(k)\,T(k,l)\,S'(l) = i S'(l) - i S'(k)$$

or equivalently

$$(l-k)_\mu\, T^\mu(k,l) = i S'^{-1}(k) - i S'^{-1}(l) \qquad (4)$$

"generalized Ward identity"

In the limit $l \rightarrow k$:

$$\Gamma^\mu(k,k) = i\frac{\partial}{\partial k_\mu} S'^{-1}(k)$$

Using $\quad iS'^{-1}(k) = \not{k} - m - \Sigma^*(\not{k})$

this gives

$$\Gamma^\mu(k,k) = \gamma^\mu + \frac{\partial}{\partial k_\mu} \Sigma^*(\not{k})$$

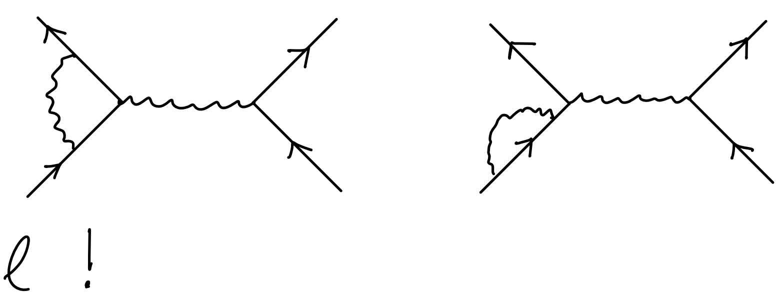
Now renormalization tells us

$$\Sigma^*(m) = 0 \quad \text{and} \quad \left.\frac{\partial \Sigma^*(\not{k})}{\partial \not{k}}\right|_{\not{k}=m} = 0$$

$$\longrightarrow \quad \bar{u}'_k \Gamma^\mu(k,k) u_k = \bar{u}'_k \gamma^\mu u_k$$

for renormalized Dirac field

This is the reason the contribution to "charge renormalization" by diagrams



cancel!

Gauge invariance

The conservation of electric charge (eq. (1)) can be used to prove gauge inv.

Consider the quantities

$$M^{\mu\mu'\cdots}_{\beta\alpha}(q, q', \cdots) = \int d^4x \int d^4x' \cdots e^{-iq\cdot x} e^{-iq'\cdot x'} \cdots$$

$$\times \langle \Psi^-_\beta | T[J^\mu(x) J^{\mu'}(x') \cdots] | \Psi^+_\alpha \rangle$$

→ matrix element of emission (absorbtion) of photons with momenta $q^\mu, q'^{\mu'}, \ldots$ in an arbitrary transition $\alpha \to \beta$

Previously, we had given a diagramatic proof that

$$q_\mu M^{\mu\mu'\cdots}_{\beta\alpha}(q, q', \cdots) = q'_{\mu'} M^{\mu\mu'\cdots}_{\beta\alpha}(q, q', \cdots)$$

$$= \cdots = 0$$

Since M is symmetric in the photon lines, it suffices to show the vanishing for contraction with q.

Proof:

Integrating by parts, we have

$$q_\mu M^{\mu\mu'\cdots}_{\beta\alpha}(q, q', \cdots) = -i \int d^4x \int d^4x' \cdots$$

$$\times e^{-iq\cdot x} e^{-iq'\cdot x'} \cdots \langle \Psi^-_\beta | \frac{\partial}{\partial x^\mu} T[J^\mu(x) J^{\mu'}(x') \cdots] | \Psi^+_\alpha \rangle \tag{5}$$

Using

$$T[J^\mu(x) J^\nu(y)] = \theta(x^0 - y^0) J^\mu(x) J^\nu(y) + \theta(y^0 - x^0) J^\nu(y) J^\mu(x)$$

we get upon taking derivatives

$$\frac{\partial}{\partial x^\mu} T[J^\mu(x) J^\nu(y)]$$

$$= \delta(x^0 - y^0) J^0(x) J^\nu(y) - \delta(y^0 - x^0) J^\nu(y) J^0(x)$$

$$= \delta(x^0 - y^0) [J^0(x), J^\nu(y)] \quad (T[\partial_\mu J^\mu(x) \cdots] = 0)$$

$(**)$

For more than two currents, we get equal time commutator for each current aside from $J^\mu(x)$ itself.

Recall

$$[J^0(\vec{x}, t), F(\vec{y}, t)] = -q_F F(\vec{x}, t) \delta^3(\vec{x} - \vec{y})$$

where

$$q_F = \sum_\ell q_\ell(\psi_\ell \in F) - \sum_{\ell'} q_\ell(\bar{\psi}_\ell \in F)$$

Moreover, $q_J = 0$ ($J^\mu(y)$ is itself neutral: $J^\mu \sim \bar{\psi}\gamma^\mu\psi$)

$$\longrightarrow \quad [J^0(\vec{x},t), J^\nu(y,t)] = 0$$

Thus (**) vanishes !

$$(5) \longrightarrow \quad q_\mu M^{\mu\mu'\dots}_{\beta\alpha}(q,q',\dots) = 0$$

$\square$

Translation invariance

The translation operator P^μ has the property

$$[P_\mu, \mathcal{O}(x)] = i\frac{\partial}{\partial x^\mu}\mathcal{O}(x)$$

where $\mathcal{O}(x)$ is any local operator

we have

$$P^\mu \psi_\alpha^+ = p_\alpha^\mu \psi_\alpha^+ , \quad P^\mu \psi_\beta^- = p_\beta^\mu \psi_\beta^-$$

$$\rightarrow (p_{\beta\mu} - p_{\alpha\mu})\langle \psi_\beta^- | T[\mathcal{O}_a(x_1)\mathcal{O}_b(x_2)\cdots] | \psi_\alpha^+\rangle$$

$$= \langle \Psi_\beta^- | [P_\mu, T[O_a(x_1) O_b(x_2) \cdots]] | \Psi_\alpha^+ \rangle$$

$$= i \left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu} + \cdots \right) \langle \Psi_\beta^- | T[O_a(x_1) O_b(x_2) \cdots] | \Psi_\alpha^+ \rangle$$

appying to

$$T[O_a(x_1) O_b(x_2) \cdots] = J^\mu(x)$$

we find

$$\langle \Psi_{\vec{p}', \sigma'} | J^\mu(x) | \Psi_{\vec{p}, \sigma} \rangle = e^{i(p'-p)\cdot x} \langle \Psi_{\vec{p}' \sigma'} | J^\mu(0) | \Psi_{\vec{p} \sigma} \rangle$$

$$\longrightarrow \langle \Psi_{\vec{p}', \sigma'} | Q | \Psi_{\vec{p}, \sigma} \rangle = \int d^3x \, \langle \Psi_{\vec{p}', \sigma'} | J^0(x) | \Psi_{\vec{p} \sigma} \rangle$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \langle \Psi_{\vec{p}', \sigma'} | J^0(0) | \Psi_{\vec{p}, \sigma} \rangle$$

$$\longrightarrow \langle \Psi_{\vec{p} \sigma'} | J^0(0) | \Psi_{\vec{p}, \sigma} \rangle = (2\pi)^{-3} q \, \delta_{\sigma' \sigma}$$